**Continuous Distributions**

* **Continuous random variable** – takes values in some interval (a, b) ⊆ R
  + A r. v. X is a function from the sample space to the real numbers
  + i.e. X : S → R
  + The range R(X) is continuous
* Individual points are intervals of length 0 – must have 0 probability; i.e.:
  + P(X=a) = 0 ∀ a ∈ R(X)
  + P(a < X < b) = P(a ≤ X ≤ b)
* **Probability density function (PDF)**
  + Assigns a probability to an x ∈ R(X)
  + f(x) ≥ 0 ∀ x ∈ R(X)
  + ∫(-∞ → ∞) f(x)dx = 1 or ∫(x ∈ R(X)) f(x)dx = 1
  + If X ~ f(x) and a < b then
    - P(a < X < b) = ∫(a → b) f(x)dx
* **Cumulative distribution function (CDF) for continuous distr.**
  + F(X) = P(X < x) = ∫(-∞ → x) f(x)dx
  + i.e. f(x) = d/dx F(x) – PDF is the derivative of the CDF
  + F(-∞) = 0
  + F(∞) = 1
* Ex. For what values of k1, k2 does the function define a PDF?
  + X ~ f(x) = {k1x2 , x ∈ (0, 1)

k2x-4 , x ∈ (3, ∞)

1. , otherwise}
   * k1, k2 ≥ 0 because f(x) ≥ 0 ∀ x
   * ∫(-∞ → ∞) f(x)dx = 1 = ∫(-∞ → 0) + ∫(0 → 1) + ∫(1 → 3) + ∫(3 → ∞)
     + = 0 + ∫(0 → 1) k1x2dx + 0 + ∫(3 → ∞) k2x-4dx
     + = k1/3 + k2/81
   * 1 = k1/3 + k2/81
   * Thus 81 = 27k1 + k2
   * 0 ≤ k1 ≤ 3, 0 ≤ k2 ≤ 81 by setting k1 & k2 each to 0

* Mean/expected value
  + E(X) = μ = ∫ x ⋅ f(x)dx
  + For a function: E(g(x)) = ∫ g(x)f(x)dx
  + For a, b ∈ R and r. v. X, Y: E(aX + bY) = aE(X) + bE(Y)
* Variance
  + Var(X) = E(X2) – (E(X))2 = ∫ x2f(x)dx – (∫ xf(x)dx)2
  + For a, b ∈ R and r. v. X, Y: Var(aX + bY) = a2Var(X) + b2Var(Y)
* **Uniform distribution**
  + Let X be a r. v. with PDF:
    - f(x) = {1/(b-a) for a ≤ x ≤ b

0 otherwise}

* + Then X ~ U(a, b)
  + F(x) = ∫(-∞ → x) f(t)dt

= {0 for x < a

(x – a)/(b – a) for a ≤ x ≤ b

1 for x > b}

* + E(X) = ∫(a → b) x ⋅ 1/(b-a) ⋅ dx = (a + b)/2 = μ
  + Var(X) = (b – a)2/12
  + Ex: if X = max {X1 … Xn}, Xi ~ U(0, 1)
    - Then FX(X) = 0 , X < 0

= xn , 0 ≤ X ≤ 1

= 1 , X > 1

* + - f(x) = FX’(X) = 0 , x < 0 and x > 1

= nxn-1 , 0 ≤ x ≤ 1

* + - E(X) = ∫(0 → 1) y ⋅ nyn-1 dy = n/(n + 1)
    - i.e. for large n, the maximum of a distribution in (0, 1) will be close to 1
* **Exponential distribution**
  + Let X ~ Poiss(λt); then the distribution of time elapsed between the occurrence of successive events is exponential
  + i.e. X ~ Exp(θ), where θ = 1/λ, θ > 0
  + Let X1 = time until the first event occurs
    - P(X1 ≤ t) = 1 – P(X1 > t) = 1 – P(no events until t) = 1 – e^(-λt)
  + X ~ Exp(λ) if
    - F(x) = 1 – e^(-λx) = 1 – e^(-x/θ) , x > 0, λ > 0
    - f(x) = λe^(-λx) = e^(-x/θ)/θ
    - E(X) = ∫(0 → ∞) x ⋅ λx^(-λx) = 1/λ = θ
    - Var(X) = 1/λ2 = θ2
  + Ex: patients at a doctor’s office is seen at a rate of 4/hour
    - λ = 4, θ = ¼
    - F(x) = 1 – e-4x
    - P(wait for more than 30 minutes) = ?
    - P(X > 0.5) = 1 – P(X ≤ 0.5)

= 1 – F(0.5)

= 1 – (1 – e-4(0.5))

= e-2

* + - P(wait for more than 1 hour given having already waited for 30 min) = ?
    - P(X ≥ 1 | X ≥ 0.5) = P(X ≥ 1 ∩ X ≥ 0.5) / P(X ≥ 0.5)

= P(X ≥ 1) / P(X ≥ 0.5)

= (1 – P(X < 1)) / (1 – P(X < 0.5)

= (1 – (1 – e-4(1))) / (1 – (1 – e-4(0.5)))

= e-2 = P(X > 0.5)

* + - i.e. memory-less property